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Subtle price discrimination and surplus extraction under uncertainty

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ABSTRACT

This paper provides a solution to Proebsting's Paradox, an argument that appears to show that the investment rule known as the Kelly criterion can lead a decision maker to invest a higher fraction of his wealth the more unfavorable the odds he faces are and, as a consequence, risk an arbitrarily high proportion of his wealth on the outcome of a single event. The paper shows that a large class of investment criteria, including 'fractional Kelly', also suffer from the same shortcoming and adapts ideas from the literature on price discrimination and surplus extraction to explain why this is so. The paper also presents a new criterion, dubbed the *doubly conservative* criterion, that is immune to the problem identified above. Immunity stems from the investor's attitudes toward capital preservation and from him becoming rapidly pessimistic about his chances of winning the better odds he is offered.

1. Introduction

Proebsting's Paradox is an argument that appears to show that the investment rule known as the *Kelly criterion* can lead a decision maker to invest a higher fraction of his wealth the more unfavorable the odds he faces are and, consequently, risk an arbitrarily high proportion of his wealth on the outcome of a single event. According to this criterion one ought to choose the size of one's investment so as to maximize the expected growth of one's wealth. Addressing the paradox is important in that it seems to contradict the well known fact that a bettor that follows the Kelly criterion can never be ruined absolutely (capital equal zero) or asymptotically (capital tends to zero with positive probability).[1] The paradox was first communicated by Todd Proebsting, a computer scientist, to Ed Thorp, a mathematician, by email, who in turn made it publicly known in an article in the September 2008 issue of Wilmott Magazine, a magazine that serves the quantitative finance community.

A second reason why addressing the paradox is important is because in recent years the Kelly criterion has developed a reputation of being a part of many successful investment strategies[2] and the claim has been made that the world's most prominent stock investor, Warren Buffett, and the world's most prominent bond investor, Bill Gross, both allocate capital in manners that are consistent with the Kelly criterion.[3] Ed Thorp himself, perhaps the main proponent of the Kelly criterion in the gambling and investment community (and a very successful hedge fund manager in his own right)[4] explicitly employs the Kelly criterion as his chief portfolio allocation rule. It has even been argued that the risk–return characteristics of the very successful investments John Maynard Keynes made on behalf of King's College Cambridge's Chest Fund from 1927 to 1945 are very similar to those generated by the Kelly criterion (Ziemba, 2005).

A third reason why addressing the paradox is important is because of the long standing (and somewhat unresolved) feud that has existed since the early 1970s between the proponents of the Kelly criterion (such as Ed Thorp and, in his time, Claude Shannon) and its opponents (prominently, Nobel Laureates Paul Samuelson and Robert Merton—among others). The opponents argue that it is irrelevant that the Kelly criterion maximizes the expected rate of growth of the individual's wealth if this criterion leads to a portfolio with risk levels that the individual is not willing to tolerate. They say that to properly advise an individual regarding portfolio choice one has to identify the individual's attitudes toward risk and construct a portfolio based on those, without regard to what the Kelly criterion would prescribe. Proponents of the Kelly criterion would respond by saying things like: "Of course this is the case, but it does

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1 See, e.g., Thorp (1969, 2008) and Breiman (1961). For a comprehensive survey on the Kelly criterion and its applications see Thorp (2006).

2 See, e.g., MacLean et al. (2010). A lay reader account of the Kelly criterion's influence on the academic and investment communities can be found in Poundstone (2005).

3 See, e.g., Thorp (2006) and Gross (2006).

4 Thorp used to run two hedge funds, Princeton–Newport Partners and Ridgeline Partners, which went nearly 30 years without a down year, and averaged 19%–20% annual returns. See Patterson (2008).

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not deny the fact that (the Kelly criterion) has an objective property: it has a better growth rate than that achieved by any other strategy" (Cover, 2008). As a response to arguments of this type Samuelson (1979) wrote a two page paper in 1979 which, in words of one syllable, argued that maximizing the expected growth rate of wealth need not be appropriate. Later on, Samuelson went on to call the Kelly criterion a "complete swindle" (Poundstone, 2005). The debate stalled more or less back then.

This paper offers some bad news and some "good news" for the proponents of the Kelly criterion. The bad news is that the problem is worse than what Proebsting and Thorp anticipated in that the following 'skimming' result can be proved: one can devise a sequence of *structured* investments, or bets, that a Kelly bettor would willingly accept that would entice the investor to risk virtually his entire wealth and that will keep the expected growth of his wealth as close to zero as one wished. The paper also shows that the so-called fractional Kelly criterion (a criterion that follows from betting only a constant fraction of what the Kelly criterion would dictate) is also vulnerable in the same way.

The "good news" is that the findings of Proebsting and Thorp, and the generalizations discussed above, cannot be used to argue against the use of the Kelly criterion because many other investment criteria, precisely all those that Samuelson and Merton would advocate,[5] also suffer from exactly the same shortcomings. Something deeper is generating these vulnerabilities, and the paper discusses what that is: the simple fact that, embedded in most reasonable betting systems one can devise lies the maxim "Good odds are worth paying for". Once this is in place, a version of the surplus-extracting *skimming* result described above basically goes through.[6]

There are exceptions, however: investors that would *very rapidly* become pessimistic about their chances of winning the better odds they are offered. The paper shows the existence of a family of investment rules with these characteristics (all 'distant cousins' of fractional Kelly). This family is somewhat immune to the *skimming* result in that there is a limit to how much those bettors will bet no matter how attractive the odds in any sequence of bets presented to them may be. This provides a half open door[7] out of the dismal world Proebsting and Thorp discovered exists for bettors who like betting, and from which they would find hard to escape. On that less pessimistic note, the paper ends.

The rest of the paper is devoted to a coherent presentation of the claims made above, together with additional commentary that would aid in the interpretation of what lies beneath Proebsting's paradox and its variants. The concluding section briefly discusses the implications of the results presented to the issues related to the structuring of financial securities, mutual fund design, and "plain vanilla" betting.

For expositional simplicity I frame all decision problems below using the terminology of *fixed-odds betting*. Bettors will be regarded as male while bookies will be regarded as female.

2. The paradox[8]

Suppose that you believe an event will occur with 50% probability and somebody offers you 2–1 odds on that event. How much money you should place on this bet depends on how you feel about the tradeoff between risk and reward that is being offered to you. If you were *neutral to risk* and all you cared about was the expected final value of wealth, then you would place 100% of your wealth on the gamble, and you would ignore the fact that you may end up losing everything with probability 50%. The Kelly criterion would have you be much more conservative than that in that it would instead have you focus on the expected growth rate of your wealth. In the case in which you are offered the 2–1 odds (call this "Situation G", for 'good'), the task is to find the fraction f^G of your wealth that solves the problem

$$\max_f \frac{1}{2}\ln(W - fW) + \frac{1}{2}\ln(W + 2fW),$$

which yields $f^G = 0.25$ and exposes you to a far lower risk[9] than if you place all your money on the bet. More generally, if you are offered a 50/50 bet that pays b to 1 the Kelly criterion would have you bet a fraction of your wealth equal to $f^* = \frac{b-1}{2b}$. Hence, if you were offered 5–1 odds ("Situation B", for 'better') according to the Kelly criterion you would place $f^B = 40\%$ on your wealth on the bet.

Now, suppose that these bets occur in sequence. You are offered 2–1 odds, bet 25% of your wealth and then are offered 5–1 odds ("Situation M", for 'mixed'). Should you place an additional bet and, if so, how much?

The Kelly criterion will indeed have you place an additional bet, which can be computed as follows:

$$\max_f \frac{1}{2}\ln(W - 0.25W - fW) + \frac{1}{2}\ln(W + 2 \times 0.25W + 5fW)$$

which yields $f^* = 0.225$.

The paradox is that the total bet in this situation, $f^M = 0.25 + 0.225 = 0.475$, is larger than the 0.4 Kelly fraction if the 5–1 odds are offered from the beginning. It is counterintuitive that you bet more when some of the bet is at unfavorable odds.[10] Todd Proebsting emailed Ed Thorp asking about this.[11]

Moreover, Thorp showed that if a gambler is offered 2–1 odds, then 4–1, then 8–1, and so on, the Kelly criterion would have you eventually bet your entire wealth, thus exposing the bettor to a risk of complete ruin of exactly 50%, just as if he was risk neutral. This appears to challenge the view commonly held of the Kelly criterion keeping the investor away from any risk of ruin.

3. The resolution

3.1. "The bettor bets more at blended 2–1 and 5–1 odds than at 5–1 odds"

While it is correct that the bettor is facing blended (average) 2–1 and 5–1 in Situation M, what matters, for the purpose of decision making is not the *average* odds but the *marginal* odds. The odds that a bettor faces determine the rate at which the individual can sacrifice money-in-the-event-of-losing to money-in-the-event-of-winning. In particular, when the bettor is offered 5–1 odds he can sacrifice one dollar when losing in exchange for five dollars when winning. The fact that the bettor already made bets at 2–1 odds does not alter the terms of the current 5–1 tradeoff. What

5 That is, all those based on *expected utility maximization*.

6 Skimming as used in the paper is a natural variant on the practices known as "price discrimination" in the economics literature. See Varian (1989).

7 'Half open' because the investors can still be skimmed, just not completely.

8 This section is based on the account of Proebsting's Paradox given in Thorp (2008) and Wikipedia (2009).

9 And a lower expected final value of your wealth.

10 Although quite obvious, it will be important in what follows to stress that f^M is the fraction of the individual's original wealth that the Kelly bettor will ultimately bet in Situation M.

11 In general, if a bettor makes the Kelly bet on a 50/50 bet with a payout of b_G, and then is offered $b_B > b_G$, the bettor will, in this situation, bet a total of

$$f^M = f^B + f^G \frac{(b_B - b_G)}{2b_B},$$

where f^i is the Kelly bet for Situation i. From this one can tell that $f^M > f^B$.

it does alter is the cash value of the bettor's wealth. This is so because individuals can now get better bets for the same cost or, equivalently, comparable bets at a lower cost than the one the bettor is now sitting on. Thus, the 2–1 bet will trade *at a discount* in an environment in which 5–1 bets are freely available. It is thus more accurate to say that in this situation the bettor faces the same (marginal) odds in Situations M and B but is *poorer* in Situation M relative to Situation B, and that is what explains the different choices in both situations. One can estimate the change in the real value of the bettor's wealth as follows: Before the individual makes any bet at 2–1 odds the individual's wealth is, say, W, all in cash. Given these odds a Kelly bettor places a bet of $0.25W$. Now consider the effect of the change in odds on the (market) value of this bet. The question is: how much cash does one have to sacrifice to obtain, at the current 5–1 odds, a bet that pays 2 when winning and -1 when losing? The answer is the value of v that satisfies

$$(2-v)+5(-1-v)=0,$$

that is, the value that makes the 2–1 bet as attractive as the newly offered 5–1 bets, or $v=-0.5$.[12] In other words, committing to a $0.25W$ bet and then seeing the odds improve to 5–1 is equivalent to not having committed to any bet at 2–1, having one's wealth change by $-0.5*0.25W=-0.125W$ and *then* facing 5–1 odds. In general, if a bettor makes the bet f^G on a bet with a payout of b_G, and then is offered $b_B>b_G$, the marked-to-market wealth W^M can be computed as follows:

$$W^M=\frac{1+\bar{b}}{1+b_B}W,$$

where $\bar{b}=f^Gb_G+(1-f^G)b_B$ is the "blended" (weighted average of) odds.[13] In the example above, the blended odds are $\bar{b}=4.25$ and thus $W^M=0.875W$.

Based on an argument along the lines of what was espoused above, Aaron Brown (in a personal communication to Ed Thorp about this paradox, see Wikipedia, 2009) argued that this analysis "makes it clear that the change in behavior [of the Kelly bettor] results from the mark-to-market loss the investor experiences when the new payout is offered". While this is absolutely correct it does not yet offer an explanation for why the mark-to-market loss makes the bettor bet more in Situation M than in Situation B. In particular, the mark-to market loss could have enticed the bettor to bet less. Why is this not the case?

3.1.1. Wealth effects of a change in odds

It turns out that a consideration of an elementary fact is an important component of the full explanation as for why the Kelly bettor bets more in Situation M than in Situation B: that the individual does not have a direct use for the bet itself, but, rather, for the outcomes that arise from betting. In other words, consider

$$\max_f \frac{1}{2}\ln(W-fW)+\frac{1}{2}\ln(W+bfW),$$

with solution $f^*=\frac{b-1}{2b}$. What contributes to the expected growth rate is not f^* directly but rather the final value of wealth in both states of nature, namely, $W-f^*W$ when losing and $W+bf^*W$ when winning. Call those wealth levels x_1^* and x_2^*, respectively. It follows that

$$x_1^*=\frac{W(1+b)}{2b},\qquad x_2^*=\frac{W(1+b)}{2}$$

and so it is clear that x_1^* and x_2^* vary directly with W,[14] which is to say that, other things equal, as the individual becomes poorer (W drops) he adjusts his possible wealth levels x_1^* and x_2^* downwards, and in this case in proportion to the decrease in W. Think of x_1^* as the dollar amount of wealth the individual keeps in cash. It is important to understand why x_1^* drops with a decrease in W. At the margin, the last dollar allocated to x_1^* and x_2^* contribute equally to the expected growth of wealth of the individual. Of course, the balance one strikes is to keep x_2 high so that one's wealth is high when one wins but to keep x_1 also high so that one's wealth is not too low when one loses. If wealth were to drop below W to, say, $\underline{W}$, but the value of x_1 were to stay at x_1^* this could only be achieved by substantially reducing the dollar amount the individual bets and thus reducing greatly the possible value of wealth when one wins, x_2. At that point the risk/reward tradeoff is such that increasing one's bet by \$1, thus reducing the value of x_1, would cost less in terms of growth rate in the event of losing than the growth rate that one gains from betting more in the event of winning.[15]

The reason why this is significant is that it was established above that the only difference between Situations M and B is that the marked-to-market wealth level of the individual is lower in Situation M than in Situation B and thus $x_1^M<x_1^B$ (and $x_2^M<x_2^B$). Therefore, if we wish to ask, as we did before, what fraction *of the original wealth W* does the individual bet in Situation M one would have to compute $f^M=\frac{W-x_1^M}{W}$ and one immediately concludes that $f^M>f^B$. This conclusion can be reached also from looking at the expression

$$f^M=\frac{W^M}{W}f^B+\left(1-\frac{W^M}{W}\right)1,$$

which says that f^M is a weighted average between f^B and one, where the weight placed on f^B is the ratio of the marked-to-market wealth to the original wealth.[16] In our example above,

$$f^M=\frac{W^M}{W}f^B+\left(1-\frac{W^M}{W}\right)=0.875*0.4+0.125=0.475,$$

as before.

To summarize, the full answer to the question "*why does the individual bet a higher fraction of W at f^M than at f^B even though he faces worse (blended) odds at M than at B*?" is as follows:

While average odds are worse at M than at B, what is important for the purpose of decision making is the marginal odds, and those are the same in both situations. After the individual bets at the original odds, b_G, his portfolio can take the values $(1-f^G)W$ and $W(1+b_Gf^G)$ with equal probability. Once the individual is offered the better odds, b_B, the exchange value of his random portfolio drops to $\frac{1+\bar{b}}{1+b_B}W$, which makes the contribution of an extra dollar of betting toward the expected growth rate of wealth larger than before, thus enticing the individual to seek some of that growth by reducing his cash levels at M relative to those held at B and thus betting more at M than at B.

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12 In general, to find the cash value of the bet $(-w_1,w_2)$ when the (market) odds are b to 1 on state 2 one simply computes the expression

$$v=\frac{b}{1+b}(-w_1)+\frac{1}{1+b}w_2.$$

What one is doing is expressing the value of the bundle $(-w_1,w_2)$ in units of the risk free asset, which is the bundle (1, 1). In this example, with $b=5$, $w_1=0.25W$ and $w_2=0.5W$ one gets $v=-0.125W$.

13 To see this notice that $W^M=W+v$ with $v=\frac{b_B}{1+b_B}\left(-f^GW\right)+\frac{1}{1+b_B}b_Gf^GW$ as in the previous endnote. The result quickly follows.

14 In economics terminology: x_1 and x_2 are "normal goods".

15 The increase in the expected growth rate from betting an extra dollar when the individual is keeping x_1^* in cash and has wealth level $\underline{W}$ is given by $\frac{1}{2}\frac{b}{\underline{W}(1+b)-bx_1^*}-\frac{1}{2}\frac{1}{x_1^*}$. It is easy to see that this expression is positive since, by construction, $\frac{1}{2}\frac{b}{W(1+b)-bx_1^*}=\frac{1}{2}\frac{1}{x_1^*}$ and $W>\underline{W}$.

16 This expression follows from the fact that $f^M=\frac{W-x_1^M}{W}$ and that $x_1^M=\left(1-f^B\right)W^M$.

4. A general phenomenon

While not negating the known optimality properties of the Kelly criterion the discussion of Proebsting's Paradox so far appears to suggest that this criterion leaves the bettor at risk of losing a substantial fraction of his wealth to a bookie (or a series of bookies) that would offer the bettor a string of bets of increasingly favorable odds. In this Section I show that, to the extent that this is so, this is also true not just for the Kelly criterion but for a wide variety of betting criteria, including fractional Kelly (the practice of betting a constant fraction of what a Kelly bettor would choose in any given situation).

Consider first the class of rules that arise as the solution to

$$\max_f \frac{1}{2}u(W - fW) + \frac{1}{2}u\,(W + bfW)\,,$$

where u is increasing, twice continuously differentiable, and strictly concave. Let $f_u\,(W, b)$ be the solution to this problem.[17] Now let us apply this decision rule in Situations G, B and M as above, so $f^G = f_u\,(W, 2)$, $f^B = f_u\,(W, 5)$, and $f^M = f^G + f^*$, where f^* solves

$$\max_f \frac{1}{2}u(W - f^G W - fW) + \frac{1}{2}u\left(W + 2f^G W + 5fW\right).$$

The question is: how does f^M compare to f^B?

Claim 1. $f^M > f^B$.

Proof. First define, as before, $x_1 = W - fW$ and $x_2 = W + bfW$ and thus rewrite the problem as

$$\max_{x_1, x_2} \frac{1}{2}u(x_1) + \frac{1}{2}u\,(x_2)$$

$$\text{subject to } \frac{b}{1+b}x_1 + \frac{1}{1+b}x_2 = W,$$

with solution $x_1^* = x_1\,(W, b)$, $x_1^* = x_1\,(W, b)$. To show $f^M = \frac{W - x_1^M}{W} > f^B = \frac{W - x_1^B}{W}$ one can focus on the comparison between x_1^M and x_1^B. In Situation M the bettor faces the same marginal odds as he faces in Situation B, but with the bettor being poorer at M. Hence, $x_1^B = x_1\,(W, 5)$ and $x_1^M = x_1\left(W^M, 5\right)$ with

$$W^M = \frac{1 + \bar{b}}{1 + b_B}W,$$

where $\bar{b} = f^A b_A + (1 - f^A) b_B$ and therefore $W^M < W$. Hence, if we can show that $\frac{\partial x_1(W,b)}{\partial W} > 0$ then it follows that $x_1^M < x_1^B$ and hence $f^M > f^B$. This turns out to be so.[18] ■

5. Fractional Kelly

As mentioned above, the fractional Kelly criterion would have you bet a fraction of what a Kelly bettor would bet in any given situation and so it can simply be described as cf^* for some constant c between zero and one, where f^* is the full Kelly fraction. Fractional Kelly, it is said, can protect the bettor from having an incorrect statistical model of the situation at hand. Can this be formalized?

In the traditional analysis of betting in which the Kelly criterion is developed there is an implicit assumption: that the odds offered to the bettor contain no information about whether one will win or lose the wager. It is not unreasonable, however, to assume that the bettor's beliefs may vary with the odds the bookie offers. For example, the bettor could wonder whether the bookie knows something he does not. After all, what else could explain him being offered such good odds? Let q be the probability the bettor places on an event taking place before he is offered any kind of bet. The bookie now offers the bettor odds b to 1 ($b > 1$). The Kelly bettor finds f to solve

$$\max_f\,(1 - q)\ln(W - fW) + q\ln\,(W + bfW)\,.$$

However, another bettor may revise his beliefs based upon the odds being offered to him. Let c be a constant between zero and one that represents his confidence in his own probabilistic estimate and let $q^c = cq + (1 - c)\,\frac{1}{1+b}$. This *cautious* bettor would pick f to solve

$$\max_f \left(1 - q^c\right)\ln(W - fW) + q^c\ln\,(W + bfW)\,.$$

The interpretation is that the bettor believes his chances of winning are worse the better the odds he is offered. Were he to have no confidence on his own estimate he would basically conclude that his probability of winning when offered b to 1 odds is $\frac{1}{1+b}$ and he would thus bet absolutely nothing.

The point here is not in the least to defend this model of belief revision. Just to indicate that this model characterizes the fractional Kelly criterion in this situation.

Claim 2. *The solution f_c to the cautious bettor problem is cf^*, where f^* is the full Kelly fraction for odds b and beliefs q.*

Proof. Consider the bet of an ordinary Kelly bettor who, for whatever reason, has beliefs over winning given by q^c. This bettor would bet a fraction of his wealth given by $\frac{q^c(1+b)-1}{b}$. Now to figure out what a cautious bettor would do simply consider this solution and replace q^c by $cq + (1 - c)\,\frac{1}{1+b}$. This yields $f_c = c\,\frac{q(1+b)-1}{b} = cf^*$. ■

5.1. Proebsting's Paradox and fractional Kelly

An advantage of this representation is that it provides a context in which we can understand how a fractional Kelly bettor would behave in Situations G, B and M as above. Consider, for example, $c = 0.5$ so $f_c^G = 0.125$ and $f_c^B = 0.2$. What about Situation M, our case with "blended odds?" Is it still so that $f_c^M > f_c^B$? It is not even clear how to apply fractional Kelly without an appeal to a representation such as the one found above. Thanks to it one can write the problem the fractional Kelly bettor faces as

$$\max_f \left(1 - q^c\right)\ln(W - 0.125W - fW)$$
$$+q^c\ln\,(W + 2 \times 0.125W + 5fW)$$
$$\text{subject to } q^c = 0.5q + 0.5\frac{1}{1+5}$$

with $q = \frac{1}{2}$ and solution $f^* = 0.125$. Hence, $f_c^M = 0.125 + 0.125 = 0.25 \geq 0.2 = f_c^B$.[19]

[17] The Kelly rule $f^* = \frac{b-1}{2b}$ is a special case from this class for $u(x) = \ln(x)$. In general, $f_u\,(W, b)$ need not be independent of W.

[18] Apply the implicit function theorem to the first order conditions for maximization of $\frac{1}{2}u(x_1) + \frac{1}{2}u\,(x_2)$ to get $\frac{\partial x_1}{\partial W} = \frac{\frac{b}{1+b}u_{22}}{\frac{1}{(1+b)^2}u_{11} + \frac{b^2}{(1+b)^2}u_{22}}$, which is greater than zero since $u_{11}, u_{22} < 0$.

[19] This conclusion could have also been reached by noticing that now $\bar{b} = 4.625$, $W^M = 0.9375W$ and by using the expression

$$f^M = \frac{W^M}{W}f^B + \left(1 - \frac{W^M}{W}\right) = 0.9375 \times 0.2 + (1 - 0.9375)$$
$$= 0.25.$$

6. What is going on?

We have seen that for all the bettors that we have considered we get that $f^M > f^B$ and are therefore vulnerable to continuing to increase their exposure to risk were they offered a string of bets of increasingly favorable odds. Why would that be so? This section provides an explanation for this.

What is important to recognize, again, is that, even though bettors do not have a direct use for the bet itself, they value betting in light of the consequences that arise from the bet. The value of a particular bet for an individual can thus be inferred from studying the criteria the individual uses for placing their bets. In particular, it turns out to be true for all of the bettors considered in this paper that the first few dollars of betting are much more valuable to the bettor than the subsequent dollars of betting.[20] Thus, a clever bookie will recognize this fact and attempt to place different bets with the bettor at different odds, depending on what the bookie perceives the value of the subsequent bets is to the bettor.[21] In principle, the bookie can extract all of the value the bettor derives from betting in this way and the bettor will likely end up betting more than if he was simply offered the most favorable odds from the string of bets in the first place. Call bets of this kind *structured bets* and, for fixed wealth W, describe them by a function $R(f)$ defined over the interval [0, 1] with the following interpretation: when the bettor bets fW dollars he gets a reward equal to $R(f)W$ dollars in the event of winning. Clearly, ordinary bets fit this formalism: in that case, $R(f) = b * f$, with b being the constant odds offered.

6.1. Skimming the Kelly bettor

Theorem 3. *Consider an event that the bettor believes has a probability of occurring equal to $\frac{1}{2}$. For any $\bar{f} \in \left(\frac{1}{2}, 1\right)$ and any (small) $\epsilon > 0$ there is a structured bet $R(f)$ such that the Kelly bettor bets a fraction $\bar{f}$ of his wealth given those odds and the expected growth rate of his wealth is ϵ.*

Proof. Consider bets of the form

$$R(f) = \begin{cases} \dfrac{1}{1-f}f & \text{if } f \leq \widehat{f} \\ \dfrac{1}{1-\widehat{f}}\widehat{f} + b\left(f - \widehat{f}\right) & \text{otherwise} \end{cases}$$

with

$$b = \left(\frac{e^{\epsilon}}{1-\bar{f}}\right)^2$$

and

$$\widehat{f} = \bar{f} - \left(1-\bar{f}\right)\frac{\sqrt{e^{2\epsilon}-1}}{e^{\epsilon}}.$$

The idea behind this bet is as follows: it offers better and better odds the more money the bettor wagers, up to $f \leq \widehat{f}$. Past that point it offers even higher (but constant) odds, given by b, on any bets placed in excess of $\widehat{f}$.

The problem for the bettor is then to

$$\max_f \frac{1}{2}\ln(W - fW) + \frac{1}{2}\ln\left(W + R(f)W\right)$$

subject to $R(f)$ as given above. First let us show that the expected growth of the bettor's wealth when betting $\bar{f}$ is indeed ϵ:

$$\frac{1}{2}\ln(W-\bar{f}W) + \frac{1}{2}\ln\left(W + \left(\frac{\widehat{f}}{1-\widehat{f}} + b\left(\bar{f}-\widehat{f}\right)\right)W\right) - \ln W$$
$$= \frac{1}{2}\ln(1-\bar{f}) + \frac{1}{2}\ln\left(1 + \left(\frac{\widehat{f}}{1-\widehat{f}} + b\left(\bar{f}-\widehat{f}\right)\right)\right)$$

but

$$\frac{\widehat{f}}{1-\widehat{f}} + b\left(\bar{f}-\widehat{f}\right)$$
$$= \frac{\bar{f}e^{\epsilon} - \left(1-\bar{f}\right)\sqrt{e^{2\epsilon}-1}}{\left(1-\bar{f}\right)\left(e^{\epsilon}+\sqrt{e^{2\epsilon}-1}\right)} + \frac{e^{\epsilon}}{\left(1-\bar{f}\right)}\sqrt{e^{2\epsilon}-1}$$
$$= \frac{\left(e^{2\epsilon} - \left(1-\bar{f}\right)\right)\left(e^{\epsilon}+\sqrt{e^{2\epsilon}-1}\right)}{\left(1-\bar{f}\right)\left(e^{\epsilon}+\sqrt{e^{2\epsilon}-1}\right)} = \frac{e^{2\epsilon}}{\left(1-\bar{f}\right)} - 1$$

and hence

$$\frac{1}{2}\ln(1-\bar{f}) + \frac{1}{2}\ln\left(1 + \frac{e^{2\epsilon}}{\left(1-\bar{f}\right)} - 1\right) = \epsilon.$$

Now notice that the expected growth rate of the bettor's wealth for $f \leq \widehat{f}$ is zero:

$$\frac{1}{2}\ln(W - fW) + \frac{1}{2}\ln\left(W + \frac{f}{1-f}W\right) - \ln W$$
$$= \frac{1}{2}\ln(1-f) + \frac{1}{2}\ln\left(1 + \frac{f}{1-f}\right) = 0.$$

This means that $f = \bar{f}$ dominates any $f \leq \widehat{f}$. Based on this the problem can now be recast as $\max_f \frac{1}{2}\ln(W - fW) + \frac{1}{2}\ln\left(W + \left(\frac{1}{1-\widehat{f}}\widehat{f} + b\left(f-\widehat{f}\right)\right)W\right)$, which, after some simplifications, reduces to

$$\max_f \frac{1}{2}\ln(1-f) + \frac{1}{2}\ln\left(1 - 2\bar{f} + f\right),$$

a problem that has the desired solution $f^* = \bar{f}$. ■

Remark 1. The condition $\bar{f} > \frac{1}{2}$ is used to guarantee that $\widehat{f} > 0$. The result is still true, and easier to prove, for smaller values of $\bar{f}$ (in those cases one may have to offer the better odds first, or offer simple, linear, odds). Those cases are not discussed in the paper as they are not as interesting as the situations in which $\bar{f}$ can be made to be as large as desired. Similar considerations apply to ϵ being "small".

6.2. Skimming the general bettor

A result analogous to the one just shown can be derived for bettors who seek to maximize

$$\max_r \frac{1}{2}u(W-r) + \frac{1}{2}u\left(W + br\right),$$

as in Section 4 above, where $r \in [0, W]$. For any such bettor with wealth W one can determine the *certainty equivalent* of his random wealth $(W - r, W + br)$ as the cash value CE that makes the bettor indifferent between the random wealth and the cash, namely, CE is the solution to

$$\frac{1}{2}u(W-r) + \frac{1}{2}u\left(W + br\right) = u(CE).$$

20 A fact known in the economics literature as the *compensated law of demand*.

21 Anyone that has been offered a deal "buy one, get one at half the price" at a store has been exposed to practices like this. Practices like this are known as *price discrimination* in the economics literature, as discussed in the introduction.

Any such bettor can be thus said to seek to maximize the certainty equivalent of his random wealth. Call those bettors the "CE" bettors.

The Kelly skimming result has the bettor subjected to an arbitrarily low growth rate of his wealth. In the case of the CE bettor consider the bettor *skimmed* when the certainty equivalent of the bettor's random wealth is arbitrarily close to the bettor's initial wealth, thus not much value has been generated for the bettor from the prospect of betting, even as the bettor willingly hands the bookie a large fraction of his wealth.[22]

Theorem 4. *Consider an event that the CE bettor believes has a probability of occurring equal to $\frac{1}{2}$. For any $\overline{r} \in (0, W)$ and any (small) $\epsilon > 0$ there is a structured bet $R(r)$ such that the CE bettor with wealth W bets $\overline{r}$ and the certainty equivalent of his portfolio is $W + \epsilon$.*

Proof. The structure of the proof is identical to the one above. See Appendix CE for the details. ■

6.3. Skimming the fractional Kelly bettor

Skimming the fractional Kelly bettor is harder than skimming an ordinary Kelly bettor since as one improves the odds offered to the fractional Kelly bettor he becomes more pessimistic about the likelihood of winning the bet. One thus has to offer more attractive odds to this bettor to entice him to give the bookie just about all his wealth.

Before showing that this can be done a modeling choice has to be made. Section 5 showed that a fractional Kelly bettor is like an ordinary Kelly bettor except for that he becomes more pessimistic about his chances of winning the more favorable the odds he is offered. In understanding how the fractional Kelly bettor behaves in the face of varying odds one must ask the question: which odds drive the behavior of the bettor's beliefs: marginal odds or average odds? Let us bias the setup against being able to prove the desired skimming result by assuming that the beliefs of the bettor are the most pessimistic given the schedule of odds offered. In other words: they will be driven by the highest marginal odds that the bettor faces.[23] As a general rule, this protects the bettor vis a vis the behavior of the ordinary Kelly bettor by reducing the desired amount the fractional Kelly bettor will wish to bet for any given set of odds and initial beliefs q. Again, the bettor's problem when he faces simple (linear) odds is to

$$\max_f \left(1 - q^c\right) \ln(W - fW) + q^c \ln\left(W + bfW\right)$$

subject to $q^c = cq + (1 - c)\frac{1}{1+b}$. In our case, in the face of the structured bet $R(f)$, the problem becomes

$$\max_f \left(1 - q^c\right) \ln(W - fW) + q^c \ln\left(W + R(f)W\right)$$

subject to $q^c = cq + (1 - c)\frac{1}{1+\overline{b}}$, where $\overline{b} = \sup_f R'(f)$. For this definition to make sense it is required for $R(f)$ to be differentiable over some interval $I \subset (0, 1)$ and for $\overline{b} < \infty$. Alternatively, that one has a way to determine what are the best incremental odds offered as part of the structured bet R. These are very weak assumptions.

Theorem 5. *Consider an event that the fractional Kelly bettor believes has a prior probability of occurring equal to $\frac{1}{2}$ and who places a confidence in his estimate given by c. For any $\overline{f} \in \left(\frac{1}{2}, 1\right)$ and any (small) $\epsilon > 0$ there is a structured bet $R^c(f)$ such that the fractional Kelly bettor bets a fraction $\overline{f}$ of his wealth given those odds and the expected growth rate of his wealth is ϵ.*

For proof see Appendix FC.

Remark 2. All the results in this paper have had as starting point prior probabilities of an event occurring equal to $\frac{1}{2}$. This is without loss of generality. Altering this number to some arbitrary number between zero and one yields essentially the same results, for suitably modified choice of odds to be offered to the bettor. This will also be true for the result that follows.

7. 'Doubly conservative' betting

What allows the fractional Kelly bettor to be skimmed in the same way that the full Kelly bettor is skimmed is the fact that, no matter how good the odds offered to the bettor are, he keeps a certain amount of optimism regarding the chances of the event in question actually taking place. In other words: even as the probability of the event taking place that is implied by the odds goes to zero the fractional Kelly's posterior beliefs remain bounded away from zero, as they approach cq from above. This does suggest a variation on the fractional Kelly criterion that would allow the implied probabilities and the posterior beliefs to converge together as the odds improve. Consider the beliefs $\overline{q}$ of an event occurring defined as follows:

$$\overline{q} = h\left(b, q\right), \quad \text{for } q \geq \frac{1}{2} \text{ and } b \geq \frac{1}{q} - 1,$$

where h is onto, decreasing in b and increasing in q.[24] Moreover, h has the following properties:

$$h\left(b, q\right) \in \left(\frac{1}{1+b}, q\right) \quad \text{for } b > \frac{1}{q} - 1$$

$$\lim_{b\to\infty} h\left(b, q\right) = \lim_{b\to\infty} \frac{1}{1+b} = 0$$

$$\lim_{b\downarrow \frac{1}{q}-1} h\left(b, q\right) = q.$$

The interpretation is as follows: q represents the bettor's prior beliefs and $\overline{q}$ represent the posterior beliefs conditional on the odds, b. As the beliefs implied by the odds offered to the bettor get close to the prior beliefs of the bettor, the posterior beliefs are also close to the prior beliefs. Also, as the beliefs implied by the odds go to zero the posterior beliefs of the bettor go to zero as well. More is needed, though: it is necessary for $\overline{q}$ to eventually approach zero at a sufficiently fast rate. A sufficient condition for this to take place is as follows:

$$h\left(b, q\right) \in o\left(\frac{1}{\ln(b)}\right).$$

Identify bettors with beliefs with all these characteristics and who seek to maximize the expected growth rate of their wealth as the "*doubly conservative*" bettors.[25]

For these bettors there is an upper bound on how much they will risk, regardless of how attractive the structure of bets offered to the bettor may be.

22 And it should be clear from the presentation that the Kelly skimming result can be recast in terms of certainty equivalents since a Kelly bettor whose wealth is expected to grow at the rate ϵ will have no problem trading his portfolio of bets for an amount of cash which is exactly equal to We^{ϵ}.

23 Marginal with respect to the bettor's dollar amount of betting. That is, if he is offered the non-linear bet $R(f)$ so that he stands to gain $R(f)W$ when betting fW, the marginal odds equal the derivative of $R(f)W$ with respect to fW, which is simply $R'(f)$.

24 Notice that h need not be continuous in either of its arguments.

25 Conservative both in his attitudes toward capital preservation and in his process of belief revision.

Theorem 6. *Consider an event that a doubly conservative bettor believes has a prior probability of occurring equal to $\frac{1}{2}$. There is $F \in (0, 1)$ such that for no structured bet $R(f)$ the bettor will risk a total fraction of his wealth greater than F on the outcome of this event.*

Proof. As first step compute the value of G^*:

$$G^* = \sup_{b \geq 1} \left(\frac{h\left(b, \frac{1}{2}\right)}{1 - h\left(b, \frac{1}{2}\right)} b \right)^{h\left(b, \frac{1}{2}\right)}.$$

This supremum exists, and that it is finite and greater to one (see Appendix DC). Next, solve for F in

$$(1 - F)\, G^* = 1.$$

To see that this F is the desired bound consider the following, so-called "two-part" structured bet, R_2: you get to bet all you wish at odds $b > 1$ provided you pay a fixed fee $T = tW > 0$ beforehand. For f to be the optimal choice of this bettor in this situation it has to be that

$$(W - tW - fW)^{(1-\overline{q})}\, (W - tW + R(f)W)^{\overline{q}} \geq W$$

and

$$\frac{1 - t + R(f)}{1 - t - f} = \frac{\overline{q}}{1 - \overline{q}} b$$

and thus

$$(1 - t - f) \left(\frac{\overline{q}}{1 - \overline{q}} b \right)^{\overline{q}} \geq 1$$

which requires

$$(1 - f) \left(\frac{\overline{q}}{1 - \overline{q}} b \right)^{\overline{q}} \geq 1.$$

Notice that this is possible only if $\left(\frac{\overline{q}}{1-\overline{q}} b \right)^{\overline{q}}$ is sufficiently high. Now, since $\left(\frac{\overline{q}}{1-\overline{q}} b \right)^{\overline{q}}$ is bounded from above by G^* this means that

$$(1 - f) \geq \frac{1}{\left(\frac{\overline{q}}{1-\overline{q}} b \right)^{\overline{q}}} \geq \frac{1}{G^*}$$

and it follows that

$$f \leq 1 - \frac{1}{G^*} = F.$$

It follows that $f > F$ cannot be implemented by using *two-part* structured bets. It turns out that this completes the proof since it is a fact that if f cannot be implemented by using *two-part* structured bets then it cannot be implemented at all. The argument is spelled out in Appendix RP. ■

Example 1. Consider a logarithmic version of the beliefs that lead to fractional Kelly:

$$\ln \overline{q} = c \ln q + (1 - c) \ln \frac{1}{1 + b}$$

for $c \in (0, 1)$, $q \geq \frac{1}{2}$ and $b \geq \frac{1}{q} - 1$. A bettor who seeks to maximize the expected growth rate of his wealth subject to these beliefs is a doubly conservative bettor as defined above. For this bettor when $q = \frac{1}{2}$ and $c = 0.5$ the bound described in Theorem 6 is given by

$$F \approx 0.19185240680033408,$$

which essentially means that this 'logarithmic fractional Kelly bettor' will never risk more than one fifth of his wealth on the outcome of a single event, no matter how attractive the structured bet offered.

The idea here, again, is not to defend this particular model of belief revision and position sizing. Rather, it is simply to point out the general features that a belief system would have to break down the almost inescapable logic underlying Proebsting's paradox, and to present a specific example (in this case, the logarithmic fractional Kelly criterion) that would implement those features.

8. Conclusions

This paper shows that the implications of the paradox identified by Proebsting and studied by Thorp and Brown run deeper than previously thought in that a wide family of betting rules also suffer from versions of the paradox. There is an underlying logic to the method one uses to generate the 'paradoxes,' one that is dependent on the simple fact that any bettor who likes betting (which is to say, for whom at least some degree of betting is instrumental in reaching whatever the bettor's goals happen to be) is willing to pay money to face sufficiently attractive odds. Once one knows the criteria the bettor uses for betting, this *willingness-to-pay* can be determined exactly, and the surplus the bettor derives from betting can be extracted from the bettor through a structured bet with the following characteristics: offer attractive odds on the condition that the bettor wagers certain amounts at less attractive odds. If the bet is designed carefully all the bettors studied in this paper (except for the so-called doubly conservative bettor) will, for better or for worse, willingly give a large fraction of their wealth to the bookies. The situation is not unlike that faced by a customer at a store that is offered a steep discount on certain items only after the customer commits to buying a certain number of those items at the "regular" price.

Having investigated how general the skimming results shown in this paper are it is refreshing to know that, from a prescriptive point of view, one can develop betting criteria that are conservative both with respect to taking excessive risk and of being excessively optimistic, so that such a doubly conservative bettor will never risk too large a fraction of his wealth on the outcome of a single event, no matter how attractive the structure of odds presented to him.

That the array of structured bets like the ones discussed above can be developed and investigated has implications in other areas, like in the design of securities, as the ideas employed in this paper could be used to create assets that would be particularly attractive to certain types of investors. Alternatively, they could be used to create the right incentives for individuals to save a given portion of their wealth in retirement funds. More plainly, they can be used to investigate the extent to which bookies in established betting markets already skim bettors using similar kinds of ideas, and the extent to which the doubly conservative criterion would protect a bettor's wealth in these real life settings. The scope of applicability of these ideas is even broader than this in the sense that they may be of value for the analysis and structuring of products in any type of market in which uncertainty plays a substantive role.

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Appendix CE. Skimming the general bettor

Consider bets of the form

$$R(r) = \begin{cases} I\,(r, W) & \text{if } r \leq \widehat{r} \\ I\,(\widehat{r}, W) + b\,(r - \widehat{r}) & \text{otherwise,} \end{cases}$$

with $I(r, W)$ implicitly defined by

$$\frac{1}{2}u(W-r)+\frac{1}{2}u\left(W+I(r,W)\right)\equiv u(W),$$

$$b=\frac{u'\left(W-\overline{r}\right)}{u'\left(W+I\left(\overline{r},W+\epsilon\right)\right)},$$

and $\widehat{r}$ is the solution for r in

$$I(r,W)=I\left(\overline{r},W+\epsilon\right)+b\left(r-\overline{r}\right),$$

where ϵ being small is used to guarantee that $\widehat{r}>0$. The problem for the bettor is then to

$$\max_r \frac{1}{2}u(W-r)+\frac{1}{2}u\left(W+R(r)\right)$$

subject to $R(r)$ as given above. Notice that R is not a fraction in this case. Rather, it is measured in dollars. By construction, for $r\leq\widehat{r}$

$$\frac{1}{2}u(W-r)+\frac{1}{2}u\left(W+R(r)\right)=u(W)$$

and $\frac{1}{2}u(W-\overline{r})+\frac{1}{2}u\left(W+R\left(\overline{r}\right)\right)=u\left(W+\epsilon\right)>u(W)$. This means that $r=\overline{r}$ dominates any $r\leq\widehat{r}$. Based on this the problem can now be recast as

$$\max_r \frac{1}{2}u(W-r)+\frac{1}{2}u\left(W+I\left(\widehat{r},W\right)+b\left(r-\widehat{r}\right)\right).$$

From the first order conditions for the maximization of this function and the definitions above one can derive the expression

$$\frac{u'\left(W-r\right)}{u'\left(W+I\left(\overline{r},W+\epsilon\right)+b\left(r-\overline{r}\right)\right)}=b,$$

and since $b=\frac{u'(W-\overline{r})}{u'(W+I(\overline{r},W+\epsilon))}$ it follows that $r^*=\overline{r}$.

Appendix FC. Skimming the fractional Kelly bettor

Consider bets of the form

$$R^c(f)=\begin{cases}(1-f)^{-\frac{1-\overline{q}}{\overline{q}}}-1 & \text{if } f\leq\widehat{f}\\ \left(1-\widehat{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}+\overline{b}\left(f-\widehat{f}\right)-1 & \text{otherwise}\end{cases}$$

where $\overline{b}$ and $\overline{q}$ solve

$$\begin{cases}\left(\dfrac{\overline{q}}{1-\overline{q}}\overline{b}\right)^{\overline{q}}=\dfrac{e^{\epsilon}}{1-\overline{f}}\\ \overline{q}=\dfrac{c}{2}+(1-c)\dfrac{1}{1+\overline{b}}\end{cases}\tag{1}$$

and $\widehat{f}$ is the solution for f in

$$(1-f)^{-\frac{1-\overline{q}}{\overline{q}}}=\left(1-\overline{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}e^{\frac{\epsilon}{\overline{q}}}+\overline{b}\left(f-\overline{f}\right).\tag{2}$$

To see that the system (1) has a solution insert the second bracketed equation inside the first and notice that as $\overline{b}$ approaches one from above the term $\frac{\overline{q}}{1-\overline{q}}$ approaches one and hence $\left(\frac{\overline{q}}{1-\overline{q}}\overline{b}\right)^{\overline{q}}$ is close to one also. On the other hand, both $\overline{q}$ and $\frac{\overline{q}}{1-\overline{q}}$ are bounded from above as $\overline{b}$ gets larger, which means that $\left(\frac{\overline{q}}{1-\overline{q}}\overline{b}\right)^{\overline{q}}$ grows without bound as $\overline{b}$ grows, eventually becoming larger than $\frac{e^{\epsilon}}{1-\overline{f}}$. The implication is that, by the intermediate value theorem, there is a value for $\overline{b}>1$ and $\overline{q}\in\left(\frac{1}{1+b},\frac{1}{2}\right)$ that satisfy both equations simultaneously. To see that Eq. (2) has a solution notice that for $f=\overline{f}$ the left hand side of (2) is smaller than the right hand side whereas for $f=\frac{\overline{f}-\overline{q}}{1-\overline{q}}$ the left hand side is positive and the right hand side is zero. Then, by the intermediate value theorem, there is a value for $f=\widehat{f}$ where Eq. (2) is satisfied. Again, $\overline{f}>\frac{1}{2}$ is used to guarantee that $\widehat{f}>0$ (and, again, a similar condition could have been derived in terms of ϵ). The rest of the proof is identical to the ones above: the problem for the bettor is to

$$\max_f (1-\overline{q})\ln(W-fW)+\overline{q}\ln\left(W+R(f)W\right)$$

subject to $R(f)$ as given above. First let us show that the expected growth of the bettor's wealth when betting $\overline{f}$ is indeed ϵ:

$$\begin{aligned}&(1-\overline{q})\ln(W-\overline{f}W)+\overline{q}\ln\left(W+\left(\left(1-\widehat{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}+\overline{b}\left(\overline{f}-\widehat{f}\right)-1\right)W\right)-\ln W\\ &\quad=(1-\overline{q})\ln(1-\overline{f})+\overline{q}\ln\left(\left(1-\widehat{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}+\overline{b}\left(\overline{f}-\widehat{f}\right)\right)\end{aligned}$$

but

$$\left(1-\widehat{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}+\overline{b}\left(\overline{f}-\widehat{f}\right)=\left(1-\overline{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}e^{\frac{\epsilon}{\overline{q}}}$$

and hence

$$(1-\overline{q})\ln(1-\overline{f})+\overline{q}\ln\left(\left(1-\overline{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}e^{\frac{\epsilon}{\overline{q}}}\right)=\epsilon.$$

Now notice that the expected growth rate of the bettor's wealth for $f\leq\widehat{f}$ is zero:

$$\begin{aligned}&(1-\overline{q})\ln(W-fW)+\overline{q}\ln\left(W+\left((1-f)^{-\frac{1-\overline{q}}{\overline{q}}}-1\right)W\right)-\ln W\\ &\quad=(1-\overline{q})\ln(1-f)+\overline{q}\ln\left((1-f)^{-\frac{1-\overline{q}}{\overline{q}}}\right)=0.\end{aligned}$$

This means that $f=\overline{f}$ dominates any $f\leq\widehat{f}$. Based on this the problem can now be recast as $\max_f (1-\overline{q})\ln(W-fW)+\overline{q}\ln\left(W+\left(\left(1-\widehat{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}+\overline{b}\left(f-\widehat{f}\right)-1\right)W\right)$, which simplifies to

$$\max_f (1-\overline{q})\ln(1-f)+\overline{q}\ln\left(\left(1-\overline{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}e^{\frac{\epsilon}{\overline{q}}}+\overline{b}\left(f-\overline{f}\right)\right).$$

From the first order conditions for the maximization of this function and the definitions above one can derive the expression

$$\frac{\left(1-\overline{f}\right)^{-\frac{1-\overline{q}}{\overline{q}}}e^{\frac{\epsilon}{\overline{q}}}+b\left(f-\overline{f}\right)}{(1-f)}=\frac{\overline{q}}{(1-\overline{q})}\overline{b},$$

and since

$$\left(\frac{\overline{q}}{1-\overline{q}}\overline{b}\right)^{\overline{q}}=\frac{e^{\epsilon}}{1-\overline{f}},$$

it follows that $\overline{f}$ is the desired solution for f.

Appendix DC. Finding G^*

Let $M(b)=h\left(b,\frac{1}{2}\right)\ln(b)$ and notice that $\lim_{b\downarrow1}M(b)=0$. Also, note that the assumption about the rate of convergence of $h\left(b,\frac{1}{2}\right)$ to zero implies that eventually h drops faster than $\ln(b)$ grows as $b\to\infty$ and therefore $\lim_{b\to\infty}M(b)=0$. This has the implication that $M(b)$ is bounded above by some number $M^*<\infty$. To see this note that for fixed $\varepsilon>0$ there is b_ε such that, for $b>b_\varepsilon$, $M(b)<\varepsilon$. So the largest value for M happens for values of b to the left of b_ε. Hence,

$$\begin{aligned}M^*&=\sup_{b<b_\varepsilon}h\left(b,\frac{1}{2}\right)\ln(b)\leq\sup_{b<b_\varepsilon}h\left(b,\frac{1}{2}\right)\cdot\sup_{b<b_\varepsilon}\ln(b)\\ &=\frac{1}{2}\ln\left(b_\varepsilon\right)<\infty.\end{aligned}$$

Now notice that $\frac{h\left(b,\frac{1}{2}\right)}{1-h\left(b,\frac{1}{2}\right)}$ is positive, decreasing in b, and no greater than one. Moreover, since $h\left(b,\frac{1}{2}\right) > \frac{1}{1+b}$ for $b > 1$ it follows that $\frac{h\left(b,\frac{1}{2}\right)}{1-h\left(b,\frac{1}{2}\right)}b > 1$ for $b > 1$. This means that $G^* > 1$. To see that $G^* < \infty$ notice that

$$e^{M(b)} = b^{h\left(b,\frac{1}{2}\right)} \geq \left(\frac{h\left(b,\frac{1}{2}\right)}{1-h\left(b,\frac{1}{2}\right)}b\right)^{h\left(b,\frac{1}{2}\right)}$$

which means that $G^* \leq e^{M^*} < \infty$.

Appendix RP. The generality of two part structured bets

The goal is to show that if one offers a doubly conservative bettor a structured bet R and he risks a fraction f^* of his wealth on the outcome of the underlying event then there exists a *two-part* structured bet R_2 such that the bettor also chooses f^* in this case. To construct R_2 first find the value b^* that satisfies

$$\frac{1+R\left(f^*\right)}{1-f^*} = \frac{h\left(\bar{b},\frac{1}{2}\right)}{1-h\left(\bar{b},\frac{1}{2}\right)}b^*,$$

where $\bar{b}$ corresponds to the highest marginal odds offered as part of R. Now find the smallest value b that satisfies

$$\frac{h\left(b,\frac{1}{2}\right)}{1-h\left(b,\frac{1}{2}\right)}b = \frac{h\left(\bar{b},\frac{1}{2}\right)}{1-h\left(\bar{b},\frac{1}{2}\right)}b^*.$$

Notice that b always exists, since h is onto and

$$\frac{h\left(\bar{b},\frac{1}{2}\right)}{1-h\left(\bar{b},\frac{1}{2}\right)}b^* \leq \frac{h\left(b^*,\frac{1}{2}\right)}{1-h\left(b^*,\frac{1}{2}\right)}b^*,$$

which implies that $1 < b \leq b^*$. Now given such b find the value of t that solves

$$\frac{1-t+bf^*}{1-t-f^*} = \frac{h\left(b,\frac{1}{2}\right)}{1-h\left(b,\frac{1}{2}\right)}b.$$

By construction, the two part structured bet defined by the pair (t, b) with the interpretation "pay tW for the right to bet as much money as you wish at odds b" is such that the bettor picks $f = f^*$.

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